Taxicab Geometry: Geometry without SAS Congruence

Undefined terms: point, line, plane, space

Incidence Axioms:

I1 Each two distinct points determines a line.
I2 Three noncollinear points determine a plane.
I3 If two points lie in a plane, then any line containing them lies in that plane.
I4 If two distinct planes meet, their intersection is a line.
I5 Space consists of at least 4 noncollinear points, and contains three noncollinear points. Each plane is a set of points of which at least 3 are noncollinear, and each line is a set of at least two distinct points.

Distance or Metric axioms:

D1 Each pair of points A and B is associated with a unique real number called the **distance** from A to B, denoted AB.
D2 For all points A and B, $AB \geq 0$ with equality only when $A = B$.
D3 For all points A and B, $AB = BA$.
D4 Ruler Postulate
The points of each line L may be assigned to the entire set of real numbers x, $-\infty < x < \infty$, called coordinates in such a manner that
(1) each point on L is assigned to a unique coordinate
(2) no two points are assigned to the same coordinate
(3) any two points on L may be assigned to the coordinate zero and a positive real number, respectively
(4) if points A and B on L have coordinates a and B, then $AB = |a - b|$.***

***THIS IS CHANGED: new formula $|x_2 - x_1| + |y_2 - y_1|$ ***

Angle Axioms:

A1 Existence of Angle Measure
Each angle $\angle ABC$ is associated with a unique real number between 0 and 180, called its measure and denoted $m\angle ABC$, No angle can have measure 0 or 180.
A2 Angle Addition Postulate
If D lies in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$.
Conversely, if $m\angle ABD + m\angle DBC = m\angle ABC$, then ray $\overrightarrow{BD}$ passes through an interior point of $\angle ABC$.

A3 Protractor Postulate

The set of rays $\overrightarrow{AX}$ lying on one side of a given line $\overleftrightarrow{AB}$, including ray $\overrightarrow{AB}$, may be assigned to the entire set of real numbers x, $0 \le x < 180$, called coordinates, in such a manner that

(1) each ray is assigned to a unique coordinate
(2) no two rays are assigned to the same coordinate
(3) the coordinate of $\overrightarrow{AB}$ is 0
(4) if rays $\overrightarrow{AC}$ and $\overrightarrow{AD}$ have coordinates c and d, then $m\angle CAD = |c - d|$.

A 4 Linear Pair Axiom: A linear pair of angles is a supplementary pair.

H1 Plane Separation Axiom

Let L be any line lying in plane P. The set of all points in P not on L consists of the union of two subsets H_1 and H_2 of P such that:

a. Each set is convex.
b. The sets are disjoint (i.e. they share no points)
c. If A is in H_1 and B is in H_2, then the segment $\overline{AB}$ intersects the line L.

Next would be SAS but it's not true in this geometry!

To model these axioms, we'll begin by choosing the Cartesian Plane with it's infinite number of point pairs. And we'll measure the distance between two points with the Taxicab Metric:

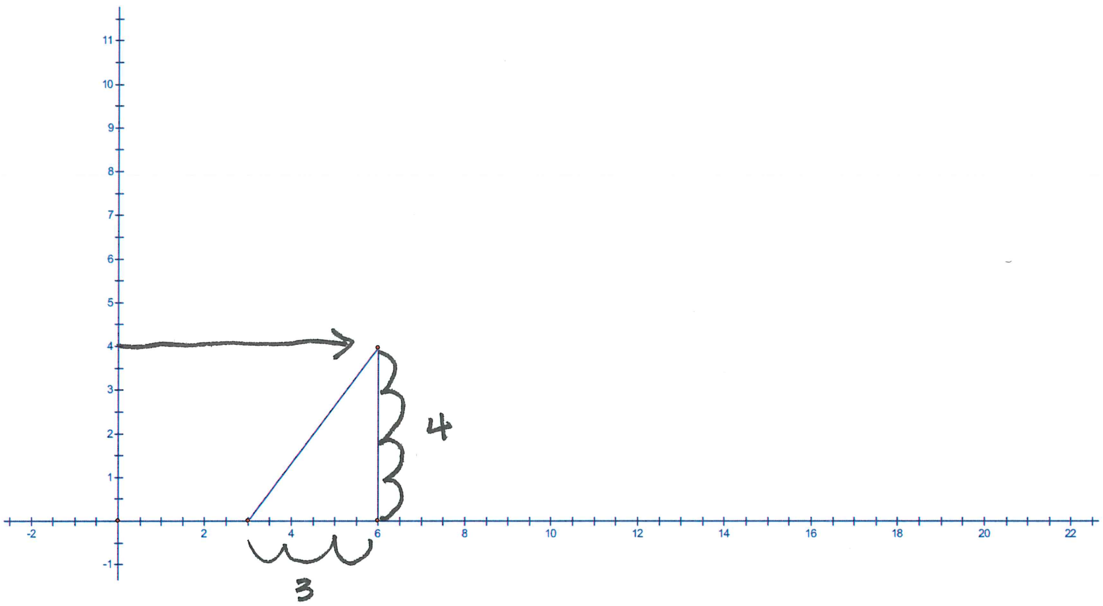
Given point X = (a, b) and point Y = (c, d).
The distance from point X to point Y, $XY = |a - c| + |b - d|$.

We will measure angles exactly the same way we've always done: with a protractor (because the Protractor Postulate is an axiom for Taxicab Geometry (TCG)).

Let's look at the graph of a triangle in the Cartesian Plane below and start talking about what's changed:

Suppose you have points: x = (3, 0), y = (6, 0), and z = (6, 4)

In the Cartesian Plane, connected by line segments, they look like this:



This picture is the same in TCG and EG (Euclidean Geometry).
And this is a right triangle in both geometries. The angle measures are exactly the same in Taxicab Geometry and in Euclidean geometry: 90, 37, and 53 degrees.
It's looking like a 3-4-5 triangle BUT that is a Euclidean name. Let's look with the Taxicab Metric and not the Euclidean Metric.

The distances between the points are NOT the same, however.

In **Taxicab Geometry**:

XY = 3, YZ = 4,

and – here's the startling one – XZ = 7. !

Check it out with the formula: x = (3, 0) and z = (6, 4) TCG distance: $|x_2 - x_1| + |y_2 - y_1|$

$$|6-3| + |4-0| = 3 + 7 = 7$$

In Euclidean Geometry we can chat casually about the Pythagorean Theorem and how the 3 – 4 – 5 right triangle illustrates it.
In Taxicab Geometry, the Pythagorean Theorem is NOT TRUE; it's not a theorem at all.

The Pythagorean Theorem is a Euclidean-specific theorem. That distance on the hypotenuse is NOT 5 it is 7 units long! Goodbye Trig. Or, rather welcome to two types of Trignonometry: EG and TCG!

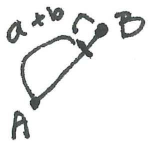
Taxicab Geometry is categorical, infinite, and non-Euclidean.

So now, both geometries have the same axioms up to and not including SAS. Both use the Cartesian Plane for points in two-dimensions. But only EG has the Pythagorean Theorem.

Betweenness: Suppose A, B, and C are distinct collinear points.

If B is between A and C, then the distance from A to B plus the distance from B to C sums to the distance from A to C. On the other hand, if the distance from AB plus the distance from B to C sums to the distance from A to C, then we know that B is between A and C. This is actually a theorem in some books!

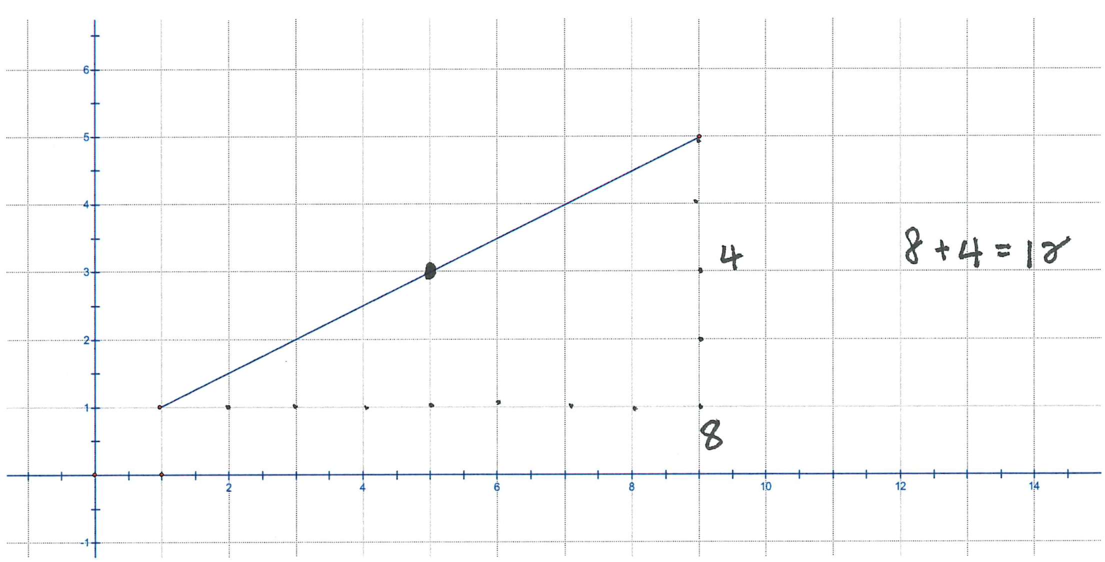
On any given line L, betweeness under the Euclidean and Taxicab Metrics coincide.



$a+b = \overline{AB}$ in both (not the same number, of course)

Well there needs to be some discussion here. This theorem doesn't say that betweeness is identical or "the same". There's some overlap in the notion in the two geometries. What didn't get mentioned is that his "betweeness" is "strictly between" in EG…and there's "metrically between" as another related notion in TCG.

Let's look at the points (1, 1) to (9, 5). How far apart are they? EG $4\sqrt{5}$ TCG 12



2 3
4 5 9
$2\frac{5}{6}$
so
$6\frac{5}{6}$ ish

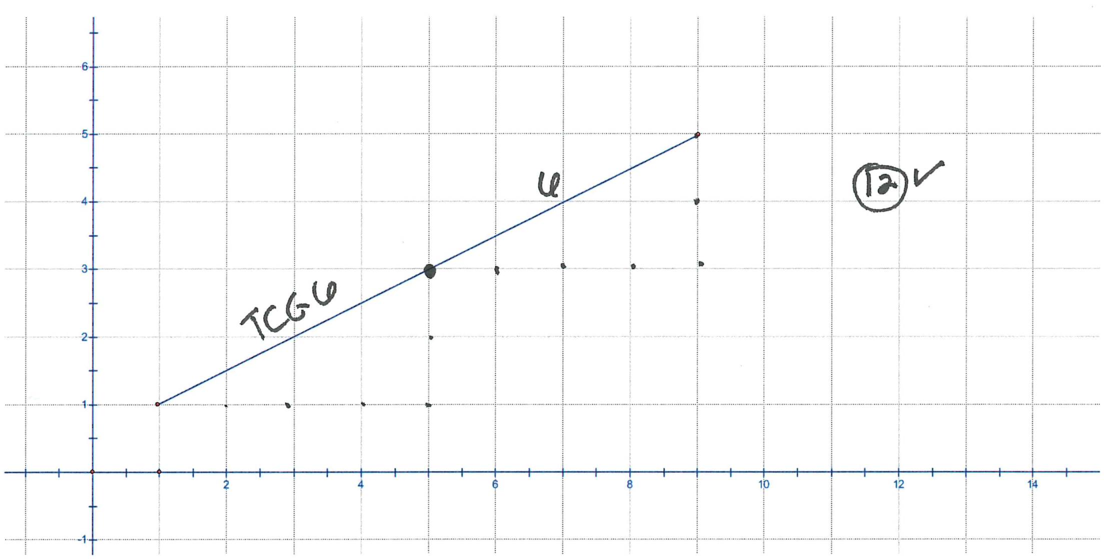
The midpoint is (5, 3). We have a unique midpoint for each line segment in both geometries. And the midpoint is strictly between the endpoints in both geometries…it's collinear.

In EG the points between the endpoints are the points on the line segment itself. These are the points that add up correctly to the Euclidean distance $4\sqrt{5}$.

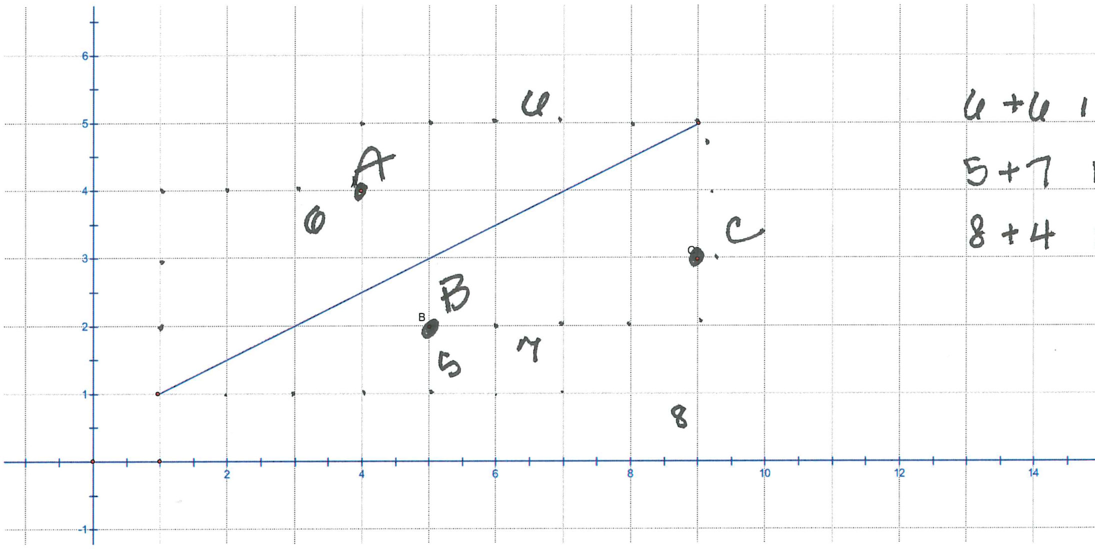
The quickest way to "measure" in TCG is to start at one point and go over and up counting unit steps as you go to the second point. The TCG distance from the two points A and B is 8 over and 4 up: 12. Points that are "between" A and B will add up to 12 when you add the distance from A to the point and from the point to B.

Let's work on a demonstration that will show a refinement to the idea of between. We're going to call this refinement "metrically between".

First, is the midpoint between the endpoints in TCG? Check: the midpoint is 4 over and 2 up from (1, 1), so it's 6 away. Then it's another 4 over and 2 up to (9, 5) for a total of 6 and these two distances sum to 12. If you check (3, 2) and (7, 4) you will find that they are between as well.



Now let's check a few other points.



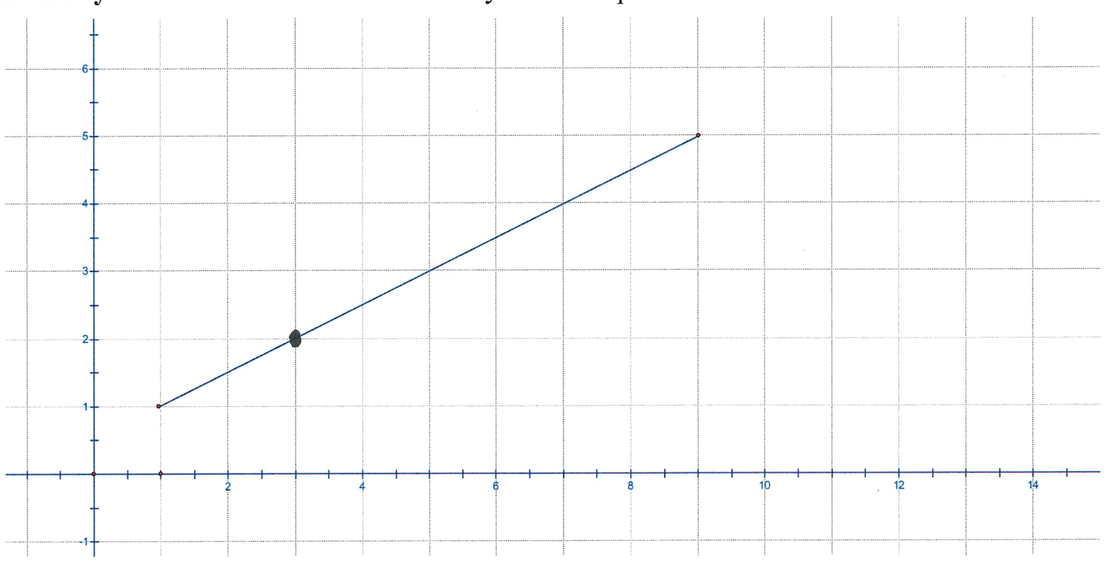
Is point A (4, 4) between the endpoints? Well from (1, 1) it's 3 over and 3 up, 6 away. And from A to (9, 5) it's 5 over and 1 up, 6 away…so it adds up correctly to 12.

Is point B (5, 2) between the endpoints? Well from (1, 1) it's 4 over and 1 up, 5 away. And from B to (9, 5) it's 4 over and 3 up, 7 away…so it adds up correctly to 12.

Is point C (9, 3) between the endpoints? Well from (1, 1) it's 8 over and 2 up, 10 away. And from C to (9, 5) it's 2 up, 2 away…so it adds up correctly to 12.

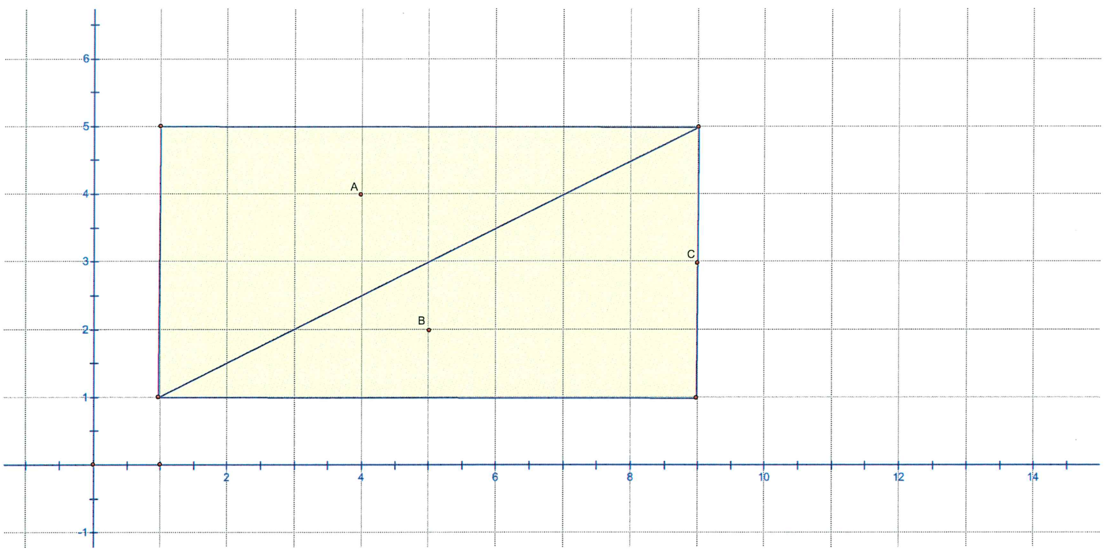
This is a situation that would NEVER, EVER happen in Euclidean Geometry. These points are technically between in the "adds up right" sense but they're NOT collinear. So we'll tweak our definition.

Metrically between: Point X is metrically between points A and B if the sum of the



distances AX + BX = AB. Note that the "collinear" statement is missing.

So, just how many of these metrically between points are there? Infinitely many. Here's a picture of where they are.

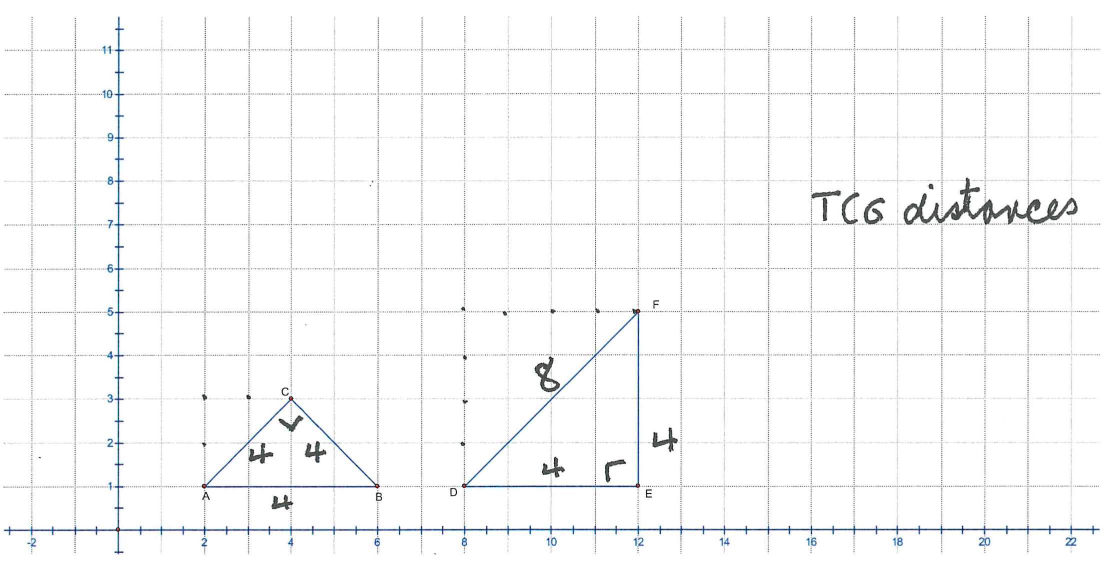


So we have betweeness in both geometries and we have the additional notion "metrically between" that is Taxicab-specific. And strict betweeness does coincide…it just got stretched a bit into this newer idea: metrically between

Another TCG truth:

In taxicab geometry, the SAS Hypothesis for two triangles does not always imply that the triangles are congruent.

All of the definitions up to this section are exactly the same. Triangles, under some correspondence, that have corresponding vertices and sides congruent are congruent. Note, though, that you have to check all 6 pieces of the correspondence in TCG



Here are two triangles: ΔABC on the left and ΔDEF on the right. They are both right triangles, no question about that. $\angle C$ and $\angle E$ are the right angles. The measure of the other two angles is 45 each. So they're both isosceles right triangles.

[Note that the sum of the interior angles for a triangle is 180 in both geometries.]

Let's set up a correspondence

$C \leftrightarrow E \qquad \overline{CA} \leftrightarrow \overline{ED}$

$A \leftrightarrow D \qquad \overline{AB} \leftrightarrow \overline{DF}$

$B \leftrightarrow F \qquad \overline{CB} \leftrightarrow \overline{EF}$

Now let's measure some side lengths.

ΔABC

In TCG: ΔABC has side length 4 and is equilateral: $AB = 4$, $AC = 4$, and $CB = 4$.

So, in TCG we have **equilateral right triangles** – an emphatically non-Euclidean situation. Go back and look more closely!

In EG: ΔABC is isosceles with legs length $2\sqrt{2}$ and base length 4

ΔDEF

In TCG: ΔDEF is isosceles with sides 4 and hypotenuse 8

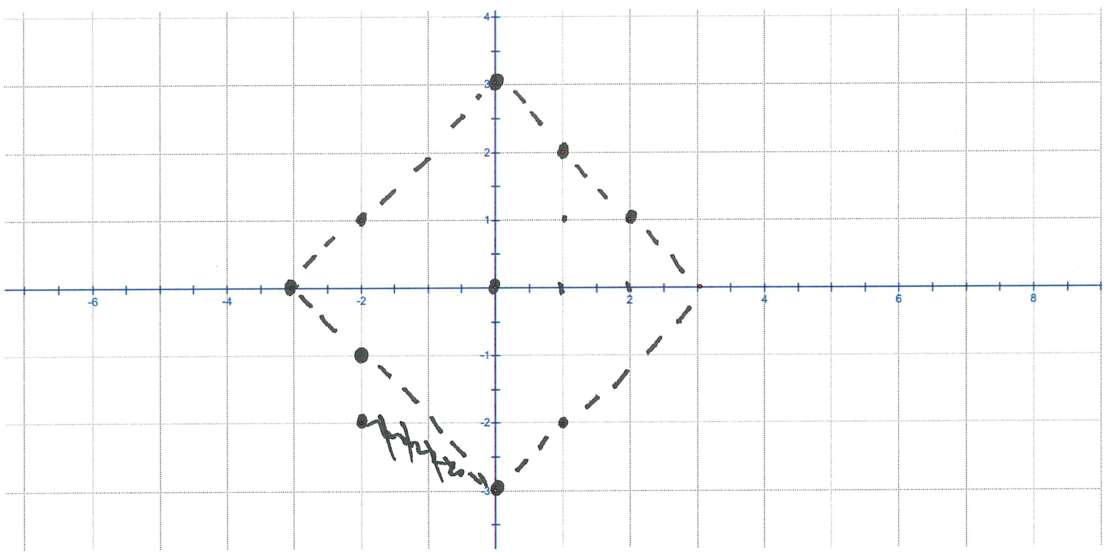
In EG: ΔDEF is isosceles with sides 4 and hypotenuse $4\sqrt{2}$

Are these two triangles congruent? Well, obviously not. The corresponding angles are congruent but the sides are not.

Here's the defining difference between TCG and EG: If we use SAS as a congruence criterion, then we could easily say that in TCG the two triangles are congruent. BOTH right angles have side lengths 4. And if you're chanting SAS, well, SAS works in one sense: 4 – 90 – 4 on ΔABC and ΔDEF, but they're really NOT congruent triangles because the hypoteni (or hypotenuses) are not congruent. So this axiom is not true in TCG. That's why we stopped where we did in those axioms.

Let's talk about Taxicab circles. We've got them in BOTH geometries and with the same definition too!

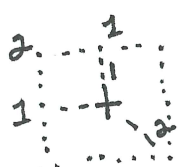
The geometric definition of a circle is: The locus of points equidistant from a given point. This definition works in EG and in TCG.

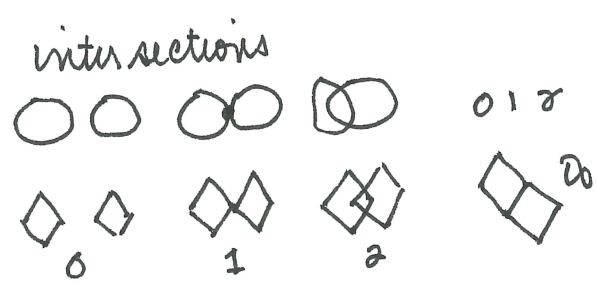


On the graph, mark off all the points that are 3 away from (0, 0). This will be the circle of radius 3, centered at (0, 0). I've put a couple of hint-points on the graph already.

Do you see a rhombus, diamond shape? Now, I contend that you've been using this Taxicab circle since you could drive. How many times have you arranged to pick up someone downtown and said "Just come out of the building and I'll pick you up. I'll circle the block until I see you". "Circle the block" is taxicab talk.

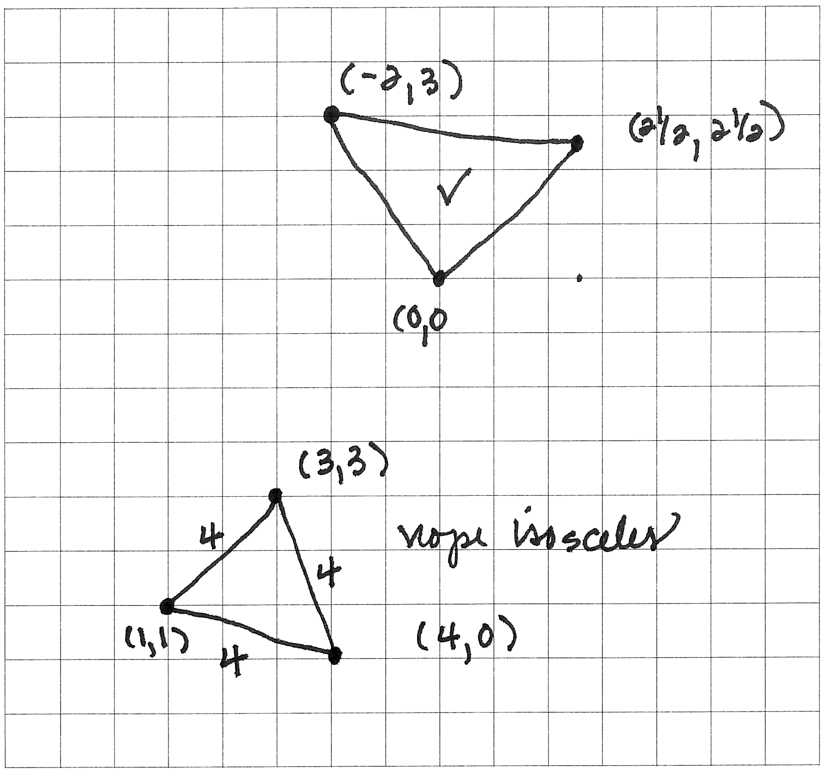
Could a square be a TCG circle too. Nope. Let's look:





So both geometries have circles and define circles in the same way. The shape of the circle and the formula for the circle are quite different, though.

Let's take an equilateral TCG triangle that is scalene in EG.



$\sqrt{8}$
$\sqrt{10}$
$\sqrt{10}$

You have a TCG triangle that is equilateral. It is, of course, scalene in EG. If you use the Law of Cosines with the Euclidean distances or an actual protractor to measure the angles, you will find that all 3 angles have different measure. So, while we do have equilateral triangles in both geometrics. The "fact" that equilateral triangles are equiangular is a Euclidean fact, not some Universal Truth kind of fact.

So here we are, with the SAME axioms up to SAS and wildly different geometries.

THE LAST ESSAY!

For the last essay write no more than TWO pages comparing and contrasting Euclidean Geometry and Taxi Cab Geometry. TCG was discovered or invented late in the 1800's by Minkowsky, a mentor of Einstein's. Be sure to cite any references or ideas from other sources!